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Wave propagation and quantum superdiffusion in a rapidly varying random potential

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We study the time evolution of a quantum particle in a rapidly varying random potential. New sets of exponent relations are found for the moments of the position of the particle. The moments exhibit weak multifractal behavior. We argue that these results are inherently quantum mechanical in nature and have no classical correspondence. Each set of exponent relations is associated with a set of conservation laws.

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I. INTRODUCTION

The behavior of a quantum particle in the presence of a random potential has attracted enormous theoretical interest in recent years. When the random potential is static the problem reduces to the well-known problem in condensed-matter physics, namely, Anderson localization [1,2]. The role of disorder on the motion of electrons or waves is well known to be crucial in low dimensions. The coherent interference effects, due to elastic scattering by static disorder, lead to strong localization of electronic eigenstates for arbitrary weak disorder in spatial dimensions $d \leq 2$. If initially the particle is placed on a given site, the long-time motion of the particle becomes subdiffusive [3]. This absence of diffusion leads to an insulating state. Several other connections of this problem to diverse fields such as physics, chemistry, and biology have been explored [4]. These studies include random multiplication and annihilation in random media (exciton trapping, chain reaction with random fissile distributions, and diffusion controlled reactions), dielectric relaxation, self-attracting polymer chains, evolution of biological species, and spin depolarization in random magnetic fields.

The case of a particle or wave motion in the presence of dynamical disorders does not seem to have drawn comparable attention. Interest in this problem began to grow only recently [5–9], even though the problem was addressed long ago [10,11]. Very recently researchers have begun to appreciate and recognize its connection to other problems in the physics of disordered systems, such as anomalous diffusion and directed polymers in random media [9]. The Schrödinger equation governing this problem is an imaginary-time version of the equation describing the directed polymer in a random potential. This problem has been intensively investigated in the past few years [12]. The Schrödinger equation in the presence of a time-dependent random potential also, to a good approximation, maps onto a problem of propagation of directed wave fronts in disordered media [9]. In a highly anisotropic medium, when the scattering potential arising due to fluctuations in the local refractive index varies slowly in one direction, one has a problem of coherent directed wave propagation in the perpendicular direction. The quantum-mechanical problem of the motion of a particle in a dynamically disordered medium is described by a time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla_d^2 \psi + V(x,t)\psi \ , \tag{1}$$

where ∇_d^2 is a d-dimensional Laplacian and $V(x,t)$ is the stochastic potential with given statistics. For the specific choice of $V(x,t)$ being Gaussian and correlated by a δ function in time, i.e.,

$$\begin{aligned} &\langle V(x,t)\rangle = 0 \ , \\ &\langle V(x,t)V(x',t')\rangle = 2V_0^2 g(x-x')\delta(t-t') \ , \end{aligned} \tag{2}$$

the problem has been solved exactly [10]. When the spatial correlation function $g(x-x')$ is a Gaussian function of its argument, the mean-squared displacement of particle $\langle x^2 \rangle$ scales as t^3. This is a superdiffusive motion at all time scales. In the presence of a parametric fluctuating potential the particle continues to absorb energy from the fluctuating force and accelerates indefinitely. In short, the particle heats up to an infinite temperature. It should also be noted that exactly the same asymptotic scaling behavior is obtained for the corresponding classical problem [10].

Golubovic, Feng, and Zeng have considered [7] classical and quantum diffusion in a time-dependent random potential with a short-range correlation both in space and time, i.e., $\langle V(x,t)V(x',t')\rangle = \exp[-a(x-x')^2 - b(t-t')^2]$. They have obtained a superlinear scaling relation for the mean-squared displacement. For a one-dimensional (1D) case, $\langle x^2 \rangle \sim t^{12/5}$ and for $d>1$, $\langle x^2 \rangle \sim t^{9/4}$. They have claimed that scaling laws are superuniversal. Recently, however, Rosenbluth [8] has obtained altogether different scaling behavior $\langle x^2 \rangle \sim t^2$ for $d>1$ for the same problems. The scaling relation in 1D is unchanged. The problem is yet to be sorted out in detail for other types of correlation functions.

It should be noted that the quantum motion of a particle on a one-band lattice or discretized version of the Schrödinger equation always gives the asymptotic behavior for $\langle x^2 \rangle \sim t$. This is a diffusive motion and does not depend on the types of correlation functions of random fluctuations. This result differs qualitatively from the continuum models and is understandable in terms of the momentum cutoff inherent in the lattice [10,11].

In an another interesting development, Bouchaud, Touati, and Sornette have studied [6] the time evolution of a wave packet in a rapidly varying random potential. They have worked with a discritized version of the Schrödinger equation. In this case the kinetic-energy operator is bounded. It is found that the wave function becomes a multifractal, i.e., it needs an infinite number of exponents to describe its evolution. The mean-squared displacement is diffusive as expected. The motion of the center of mass of the wave packet is characterized by a new exponent, $\langle [x^2] \rangle \sim t^{2\nu}$, with $\nu \approx \frac{1}{4}$. This subdiffusive motion is in agreement with the earlier prediction [13].

Here we analyze the quantum motion in a continuum with statistics of potential fluctuations given by Eq. (2). We also give details of our earlier calculations. We show here that the scaling relations for moments of displacement $\langle x^n \rangle$ depend sensitively on the nature of $g(x-x')$. In fact we can obtain infinite set of new scaling exponents. Each set of exponents is associated with certain conservation laws associated with the kinetic energy. These conservation laws are unique to quantum systems. We also believe that the same result cannot be obtained for a corresponding classical problem.

II. THEORETICAL RESULTS

To this end we will now obtain an exact solution of the quantum problem on a continuum. For simplicity we shall treat the case of one space dimension. Generalization to arbitrary dimension is straightforward as the following treatment shows. Our scaling relations are independent of dimensionality. First we will set up an equation for the density matrix. All the physical quantities of interest can be conveniently expressed in terms of an averaged reduced density matrix $\langle \rho(x',x,t) \rangle$, where

$$\rho(x',x,t) = \psi^*(x',t)\psi(x,t) \tag{3}$$

and the angular brackets denote the average over the realization of the stochastic potential $V(x,t)$. Using Eqs. (1) and (3), the equation of motion of the averaged density matrix can be written as

$$\frac{d}{dt}\langle \rho(x',x,t) \rangle = -\frac{i\hbar}{2m}\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2}\right]\langle \rho(x',x,t) \rangle + \frac{1}{\hbar}\langle V(x,t)\rho(x',x,t) \rangle - \frac{1}{\hbar}\langle V(x',t)\rho(x',x,t) \rangle \ . \tag{4}$$

The occurrence of $\langle V(x,t)\rho(x'x,t) \rangle$ on the right-hand side would normally lead to a hierarchy of coupled equations. The choice of Gaussian disorder, however, enables us to obtain a closed set of equations for the density matrix $\langle \rho(x',x,t) \rangle$. The Gaussian choice leads to a factorization of the averages of the form $\langle V(x,t)\rho(x',x,t) \rangle$ as we shall see now.

Clearly, $\rho(x',x,t)$ is a functional of the Gaussian random variables $V(x,t)$, and hence the Novikov theorem [14] for the functions of Gaussian random variables applies, namely,

$$\langle V(x,t)\rho(x',x,t) \rangle = \int_{x''}\int_{t''} \langle V(x,t)V(x'',t'') \rangle \left\langle \frac{\delta\rho(x',x,t)}{\delta V(x'',t'')\partial x''\partial t''} \right\rangle dx''dt'' \ , \tag{5}$$

where $\delta\rho(x',x,t)/[\delta V(x'',t'')\partial x''\partial t'']$ is a functional derivative of ρ with respect of V. From Eqs. (2) and (5), we get

$$\langle V(x,t)\rho(x',x,t) \rangle = V_0^2 \int g(x-x'') \left\langle \frac{\delta\rho(x',x,t)}{\delta V(x'',t'')\partial x''\partial t''} \right\rangle dx'' \ . \tag{6}$$

Substituting Eq. (6) into Eq. (5) we get

$$\frac{d}{dt}\langle \rho(x',x,t) \rangle = -\frac{i\hbar}{2m}\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2}\right]\langle \rho(x',x,t) \rangle + \frac{1}{h}V_0^2 \int g(x-x') \left\langle \frac{\delta\rho(x',x,t)}{\partial V(x'',t'')\partial x''\partial t''} \right\rangle dx'' - \frac{1}{h}V_0^2 \int g(x'-x'') \left\langle \frac{\delta\rho(x',x,t)}{\partial V(x'',t'')\partial x''\partial t''} \right\rangle dx'' \ . \tag{7}$$

Now, integrating Eq. (4) with respect to t before averaging gives

$$\rho(x',x,t) - \rho(x',x,t=0) = -\frac{i\hbar}{2m}\int_0^t \left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2}\right]\rho(x',x,t')dt' + \frac{i}{\hbar}\int_0^t V(x,t')\rho(x',x,t')dt' - \frac{i}{\hbar}\int_0^t V(x',t')\rho(x',x,t')dt' \ . \tag{8}$$

Let us take a functional derivative of $\rho(x',x,t)$ with respect to $V(x'',t)$. From Eq. (8) it can at once be written as

$$\frac{\delta\rho(x',x,t)}{\delta V(x'',t'')\partial x''\partial t''} = \frac{i}{\hbar}\int_0^t \delta(x''-x)\delta(t-t')\rho(x',x,t')dt' - \frac{i}{\hbar}\int_0^t \delta(x''-x')\delta(t-t'')\rho(x',x,t')dt' = \frac{i}{2\hbar}\delta(x''-x)\rho(x',x,t) - \frac{i}{2\hbar}\delta(x''-x')\rho(x',x,t) \ . \tag{9}$$

Substituting Eq. (9) into Eq. (7) we get

$$\frac{d}{dt}\langle \rho(x',x,t)\rangle = -\frac{i\hbar}{2m}\left[\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial x'^2}\right]\langle \rho(x',x,t)\rangle + \frac{i}{\hbar}V_0^2\int g(x-x'')\left[\frac{i}{2\hbar}\delta(x''-x)\langle \rho(x',x,t)\rangle - \frac{i}{2\hbar}\delta(x''-x')\langle \rho(x',x,t)\rangle\right]dx'' - \frac{i}{\hbar}V_0^2\int g(x'-x'')\left[\frac{i}{2\hbar}\delta(x''-x)\langle \rho(x',x,t)\rangle - \frac{i}{2\hbar}\delta(x''-x')\langle \rho(x',x,t)\rangle\right]dx'' . \quad (10)$$

Simplifying Eq. (10), we get

$$\frac{d}{dt}\langle \rho(x',x,t)\rangle = -\frac{i\hbar}{2m}\left[\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial x'^2}\right]\langle \rho(x',x,t)\rangle - \frac{V_0^2}{\hbar^2}[g(0)-g(x-x')]\langle \rho(x',x,t)\rangle . \quad (11)$$

The above equation is a closed equation for the single-particle density matrix. Earlier, Madhukar and Post [15] obtained a closed equation for the density matrix for the motion of a quantum particle on a lattice with a site-diagonal and nearest-neighbor off-diagonal dynamic disorder. One obtains such a closed-form equation by virtue of the white-noise nature of the random potential. For a general non-white-noise random potential instead one obtains a hierarchy of coupled equations. Equation (11) has to be solved subject to the initial condition that the particle was "prepared" initially in a wave packet centered at the origin $x=0$. We shall take conveniently

$$\rho(x',x,t=0)=\psi^*(x',t=0)\psi(x,t=0) ,$$

where

$$\psi(x,t=0)=\frac{1}{(2\pi)^{1/4}\sigma^{1/2}}e^{-x^2/4\sigma^2} . \quad (12)$$

This ensures correct normalization, $\int_{-\infty}^{+\infty}\rho(x,x,t=0)dx=1$. Here, σ denotes the spatial spread of the initial wave packet. Because of the unbounded nature of the kinetic-energy operator in the continuum limit, it is necessary to choose a wave packet with $\sigma>0$. The asymptotic ($t\to\infty$) behavior is, of course, independent of the precise form of the wave packet. This problem does not arise in the case of the lattice Hamiltonian H_L, which is bounded. Equation (12) can be solved by first taking the time Laplace transform and then considering the resulting hyperbolic equation in the two independent variables x and x'. We get

$$\frac{2i\hbar}{m}\frac{\partial^2}{\partial X\,\partial Y}\tilde{R}(X,Y,s)+\left[s+\frac{V_0^2}{\hbar^2}g(0)-\frac{V_0^2}{\hbar^2}g(Y)\right]\tilde{R}(X,Y,s)=R(X,Y,t=0) , \quad (13)$$

where we have introduced the characteristic coordinates $X=x+x'$, $Y=x-x'$. Here, s is the Laplace transform variable. We have defined

$$R(X,Y,t)=\rho(x',x,t) ,$$
$$\tilde{R}(X,Y,s)=\int_0^{\infty}R(X,Y,t)e^{-st}dt ,$$

with

$$\bar{R}(X,Y,t)=\int_{-\infty}^{+\infty}R(X,Y,t)e^{iKX}dX . \quad (14)$$

Here the overbar denotes the spatial Fourier transform, while the tilde denotes the time Laplace transform.

Equation (13) can be converted into an ordinary first-order differential equation in Y by taking the Fourier transform with respect to X, which can then be solved readily subject to the initial condition to give

$$\tilde{\bar{R}}(K,Y=0,s)=\int_0^{\infty}\{2\exp\{-[2\sigma^2+(\hbar^2Y^2/2m^2\sigma^2)]K^2\}e^{-sY}\exp\left\{-\frac{V_0^2}{\hbar^2}\int_0^{Y}\left[g(0)-g\left[\frac{2\hbar|K|Y'}{m}\right]\right]dY'\right\}dY . \quad (15)$$

The right-hand side of this equation is already in the form of a Laplace transform. Hence, on inversion, we get at once

$$\bar{R}(K,Y=0,t)=2\exp\{-[2\sigma^2+(\hbar^2t^2/2m^2\sigma^2)]K^2\}\exp\left\{-\frac{V_0^2}{\hbar^2}\left[g(0)t-\int_0^{t}g\left[\frac{2\hbar|K|Y'}{m}\right]dY'\right]\right\} . \quad (16)$$

Equation (16) is identical to Eq. (15) of Ref. [10]. The K dependence of Eq. (16) depends on the form chosen for the spatial correlation function g. In fact, we will show in the following that the scaling relations for the position moments depends sensitively on the form of g. In Ref. [10] the scaling relations have been obtained only for a specific dependence of g on the spatial coordinate. One readily confirms that Eq. (16) fulfills the normalization and the initial condition.

The mean-squared displacement can be expressed as

$$\langle x^2(t)\rangle = -\frac{1}{8}\frac{\partial^2}{\partial K^2}\bar{R}(K,Y=0,t)\bigg|_{K=0}, \tag{17a}$$

and other higher moments

$$\langle x^4(t)\rangle = \frac{1}{32}\frac{\partial^4}{\partial K^4}\bar{R}(K,Y=0,t)\bigg|_{K=0}, \tag{17b}$$

$$\langle x^6(t)\rangle = -\frac{1}{128}\frac{\partial^6}{\partial K^6}\bar{R}(K,Y=0,t)\bigg|_{K=0}, \tag{17c}$$

etc. Equation (17) holds provided $\tilde{R}(K,Y=0,t)$ is analytic in K around $K=0$.

Before proceeding further we would like to make some important observations. One can readily verify from the master equation (11) that $(d/dt)(\mathrm{Tr}\rho^2)<1$. This implies that under the dynamical evolution, quantum-mechanical pure states are transformed asymptotically into a statistical mixture. This is due to the fact that the fluctuating potential dephases the quantum evolution [16] and leads to irreversible behavior. The statistics of random potential is identical under time reversal, e.g., $\langle V(x,t)\rangle = \langle V(x,-t)\rangle$. In fact, we can assume $V(x,t)=V(x,-t)$; this does not effect any property of the dynamical evolution. This is because the dynamical evolution up to time t contains only information about the random potential from time 0 to time t. We have set initial time $t=0$. Now one can notice from the Eq. (4) that the behavior of the density matrix is invariant under time reversal. Hence if $\rho(x',x,t)$ is a solution then $\rho^*(x',x,-t)$ is also a solution of Eq. (4). Since ρ and ρ^* contain the same information, if we start with an equation for $\rho^*(x',x,t)$ and, after averaging over random potential, let $t\to -t$, we arrive at another solution to the same physical problem. The final required second solution turns out to be same as that of Eq. (15) except the sign in front of V_0^2 is changed. It turns out that in real situations one of the solutions is physically relevant. In fact, one can notice that for a given $g(x-x')$ one of the solutions gives in the asymptotic time domain negative values for expectation values of even powers of x, and hence that solution should be rejected.

We first consider a case where the spatial part of the potential correlation function is given by

$$g(y) = \frac{1}{(2\pi)^{1/2}\alpha}\exp[-y^2/2\alpha^2] \ . \tag{18}$$

The mean-squared displacement, which has been worked out earlier [10], is given by

$$\langle x^2(t)\rangle = \sigma^2 + \frac{\hbar^2}{4m^2\sigma^2}t^2 + \frac{1}{3\sqrt{2\pi}}\frac{V_0^2}{m^2\alpha^3}t^3$$
$$\approx \frac{1}{3\sqrt{2\pi}}\frac{V_0^2}{m^2\alpha^3}t^3 \quad \text{for } t\to\infty \ . \tag{19}$$

This is an exact result. It shows clearly that the particle motion is nondiffusive on any time scale. It can be easily shown that asymptotically all the higher moments $\langle x^{2n}\rangle$ scale as t^{3n}. It is important to note here that the coefficients in front of t^{3n} do not involve the Planck constant $\hbar$. This implies that we would have obtained exactly the same asymptotic behavior by doing a classical counterpart; quantum effects for continuous potentials such as considered here are only relevant at very early times. With this choice for $g(y)$, the system steadily gains energy at a constant rate. The rate of change of average kinetic energy $H=P^2/2m[\equiv -(\hbar^2/2m)(\partial^2/\partial x^2)]$,

$$\frac{\partial}{\partial t}\langle H\rangle = \mathrm{Tr}\left[\frac{P^2}{2m}\frac{\partial\rho}{\partial t}\right] . \tag{20}$$

Now using Eq. (11), the only nonvanishing contribution comes from the last term in (11) and equals $(V_0^2/\hbar^2)\int[(\partial^2/\partial x^2)g(x-x')\rho(x,x',t)]_{x'=x}dx$. Integrating by parts, we finally get [17]

$$\frac{d}{dt}\langle H\rangle = \frac{V_0^2}{2m}g''(0) \ . \tag{21}$$

The right-hand side of Eq. (21) is a constant and is an energy rate constant.

Next we consider a correlation function given by $g(y)=\exp[-\alpha y^4]$. For this case first one can readily show that the average energy is a constant of motion, i.e., $d\langle H\rangle/dt=0$ [see Eq. (21)]. Note that the energy of the system is not strictly conserved ($\langle H^2\rangle$ is not a constant of motion), but the energy is conserved on the average. Such a conservation law does not arise in the classical counterpart. One can readily verify this, starting with a classical equation of motion $m\,d^2x/dt^2=V(x,t)$. The kinetic energy is given by $\frac{1}{2}(dx/dt)^2$. We have set the unit of mass m equal to unity. For the kinetic-energy conservation on an average we require that $\frac{1}{2}\int_0^t\int_0^t\langle V(x(t'),t')V(x(t''),t'')\rangle dt'dt''$ be constant independent of space and time. Given the correlation function for random potential [Eq. (2)], the average kinetic energy is given by $V_0^2 g(0)t/2$. This shows that in a classical treatment a particle on an average gains the kinetic energy at a constant rate. After explaining the quantum nature of the conservation law, let us look at the effect of this on the behavior of moments of position of the quantum particle. The mean-squared displacement is given by

$$\langle x^2\rangle = \sigma^2 + \frac{\hbar^2}{4m^2\sigma^2}t^2 \tag{22a}$$

and the fourth moment $\langle x^4\rangle$ is given by

$$\langle x^4\rangle = 2\sigma^2 + \frac{\hbar^2 t^2}{m^2} + \frac{\hbar^4 t^4}{8m^2\sigma^4} + \frac{24}{5}\frac{V_0^2\hbar^2\alpha^2}{m^4}t^5$$
$$\approx \frac{24}{4}\frac{V_0^2\hbar^2\alpha^2}{m^4}t^5 \quad \text{for } t\to\infty \ . \tag{22b}$$

Similarly,

$$\langle x^6 \rangle \approx \frac{72}{5} \frac{V_0^2 \alpha \hbar^4}{m^6 \sigma^2} t^7 \quad \text{for } t \to \infty \ , \tag{22c}$$

and other moments scale as $\langle x^8 \rangle \sim t^{10}$, $\langle x^{10} \rangle \sim t^{12}$, $\langle x^{12} \rangle \sim t^{15}$, etc. One can write these asymptotic results in a simple form,

$$\langle x^{2n} \rangle \sim \begin{cases} t^2, & n \le 1 \\ t^{(5/2)n}, & n > 1 \text{ and } n \text{ even} \\ t^{(5/2)(n-1)+2}, & n > 1 \text{ and } n \text{ odd} \ . \end{cases} \tag{23}$$

Unlike the case $g(y) \sim e^{-\alpha y^2}$, all the coefficients in front of asymptotic time dependence involve the Planck constant [see, for example, Eqs. (22b) and (22c)]. This shows that these new relations arise essentially in a quantum treatment and are associated with the conservation laws mentioned above. None of the moments $\langle x^{2n} \rangle$ can be represented by a single functional from $t^{\tau(2n)}$, with a unique function τ. We need three different types of τ function. Hence our quantum dynamical process leads to a weak multifractal nature for the moments. We have used the word weak in the sense that for actual multifractal nature one requires an infinite hierarchy of exponents.

We can generalize our results to other simple analytical correlation functions (analytic at the origin) like $g(y) = \exp[-\alpha y^{2m}]$ and $m > 1$. The asymptotic scaling relations for moments are given by

$$\langle x^{2n} \rangle \sim \begin{cases} t^{2n} & \text{for } n \le m \\ t^{[(2m+1)/m]n} & \text{for } n > m \end{cases} \tag{24}$$

and integral multiples of m. For successive n between two integer numbers am and $(a+1)m$ (where a is an integer) the exponents vary in steps of 2, starting with a lowest value of $(2m+1)a$. As in the earlier situation, all the coefficients contain $\hbar$. For the above case of the correlation function we have $(m-1)$ conservation laws, namely, $\langle H \rangle$, $\langle H^2 \rangle, \ldots, \langle H^{m-1} \rangle$ are conserved. However, higher moments $[>(m-1)]$ are not conserved.

III. DISCUSSION

We have analyzed a quantum motion of a particle subject to a stochastic potential, where the correlation function is correlated by a δ function in time but arbitrarily correlated in space. The problem has been solved analytically. In our analysis we have considered spatial correlations of the form $g(y) \sim e^{-\alpha y^{2m}}$ which is analytic at $y=0$. We have shown that moments exhibit a weak multifractal behavior. The exponents are not universal and depend on the value of m. This rules out superuniversality of exponents for this problem. For $m > 1$, the behavior is due purely to quantum nature. This is somewhat counterintuitive from the fact that stochastic noise would have dephased the quantum evolution leading to a classical behavior at large times. However, for $m > 1$ we have shown that there arises $(m-1)$ conservation laws, namely, moments of energy up to $(m-1)$ are conserved on the average for all times. These quantum-mechanical conservation laws constrain the quantum evolution for all times leading to new features not contained in a classical counterpart. None of the moments with power less than or equal to $2(m-1)$ feel the effect of dynamical disorder and consequently evolve ballistically. However, the effect of dynamical disorder is seen in the moments with power greater than $2(m-1)$. This is a very special type of motional narrowing effect.

We believe that a new class of exponent relations may arise, if we consider correlation functions $g(y)$ nonanalytic at $y=0$. These special cases call for a somewhat detailed evaluation. However, all other types of analytical correlation functions will lead to similar behavior with some variations. For example, all analytical correlation functions $g(y)$ which can be expanded into same powers of y belong to the same universality class. In this class we have the same set of energy-conservation constraints. Our analysis also lead to a conclusion that there are infinitely many types of stochastic potentials [e.g., different correlation function $g(y)$] that lead to the same constraints on moments of energy. Conversely, with given constraints for quantum evolution, the underlying stochastic process for potential is not unique.

In our analysis we have been restricted to integer moments. At present it is not clear whether scaling properties of fractional exponents have simpler relations. It will be also interesting to explore the relationship between the multifractal behavior of the wave function to that of multifractal behavior of moments. On a lattice, discretization will lead to a momentum cutoff and therefore does not exhibit the behavior obtained in continuum theories. On a lattice, the quantum motion of a particle gives asymptotic scaling behavior, and the particle will always propagate diffusively. However, for short-time scales, motion on a lattice should exhibit the behavior obtained in a continuum version. Further research along these lines is in progress.

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